

Bohm's approach and individuality.

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Abstract

Ladyman and Ross (LR) 2007 argue that quantum objects are not individuals (or are at most weakly discernible individuals) and use this idea to ground their metaphysical view, ontic structural realism, according to which relational structures are primary to things. LR acknowledge that there is a version of quantum theory, namely the Bohm theory (BT), according to which particles do have definite trajectories at all times (Bohm 1952; Bohm and Hiley 1993). This would suggest that quantum particles are individuals after all, with position being the property in virtue of which particles are always different from one another. However, LR refer to research by Brown *et al.* (1996) which they interpret as saying that in BT, the properties normally associated with particles (mass, charge, etc.) are inherent only in the quantum field and not in the particles (in BT it is assumed that a particle is always accompanied by a quantum field). It would then seem that there is nothing there in the trajectories unless one assumes the existence of some “raw stuff” of the particle. In other words it seems that *haecceities* are needed for the individuality of particles of BT, and LR dismiss this as idle metaphysics. In this paper we point out, following Brown *et al.*(1996, 1999) that it is reasonable to assume that in BT properties such as mass and charge *also* reside in the particles (the principle of generosity). Thus, if BT is correct, quantum objects might be individuals after all. However, we move on to emphasize that Bohmian quantum individuals, while in some ways similar to classical particles, also differ from these radically. We will discuss this issue in

the light of new developments in the underlying mathematical structures, due to de Gosson and Hiley. In particular, we will show how the mathematical structure of the double cover of the underlying symmetry groups help to understand the relation between classical dynamics and quantum dynamics, as well as the similarities and differences between classical and quantum individuals. We conclude that while BT enables us to retain the notion of individuals in non-relativistic quantum theory, these individuals are very different from those of classical physics. It is likely that they can be best understood in the context of a structuralist, process-oriented view, such as Bohm and Hiley's broader implicate order framework. Thus, while we think that the prospects of individuality in quantum theory are stronger than what LR imply, we agree with them that structuralist considerations are important in fundamental physics more generally.

The usual interpretation of the quantum theory implies that we must renounce the possibility of describing an individual system in terms of a single, precisely defined conceptual model. We have, however, proposed an alternative interpretation ...which leads us to regard a quantum-mechanical system as a synthesis of a precisely definable particle and a precisely definable ψ -field... (Bohm 1952a: 188)

1 Introduction.

Perhaps the greatest challenge to the notion that objects are individuals with well-defined identity conditions comes from modern quantum and relativity physics. For, ever since the early days of the quantum revolution, the identity and individuality of quantum systems has frequently been called into question (see e.g. French 2011 and the references therein; French and Krause 2006; Ladyman and Ross 2007, ch 3).

Many of the founding figures of quantum theory, and most notably Niels Bohr, held that it is not possible to describe individual quantum objects and their behaviour in the same way as one can in classical physics. Quantum phenomena are often thought to be holistic in such a way that we have to be careful when applying our common sense notions of individual objects when speaking of quantum objects such as electrons (for a recent penetrating discussion of the Bohr approach, see Plotnitsky 2010). The idea that quantum objects might, in some sense, be "non-individuals" was also considered early on by, for example, Born, Heisenberg and Weyl (French 2011: 6).

One of the physicists who almost throughout his career emphasized the

wholeness of quantum phenomena was David Bohm (1917-1992). For example, in his 1951 text-book “Quantum Theory”, he characterized individual quantum objects in strongly relational and contextual terms:

...quantum theory requires us to give up the idea that the electron, or any other object has, by itself, any intrinsic properties at all. Instead, each object should be regarded as something containing only incompletely defined potentialities that are developed when the object interacts with an appropriate system (1951: 139).

His work on the implicate order, which begins to develop in the early 1960s, aims to develop a deeper underlying theory from which quantum theory and relativity can be derived as approximations, and their relation thus understood. This framework suggests a strongly structuralist, process-oriented way of understanding individual quantum systems. At an early phase of this work Bohm wrote:

In this theory ... the notion of a separately existing entity simply does not arise. Each entity is conceptually abstracted from a totality of process... with the electron, what actually exists is a structure of underlying elementary processes or linkages supporting a pattern corresponding to an electron. (1965a: 291).

In the later implicate order view, an electron is not a little billiard ball that persists and moves, but should more fundamentally be understood as

...a recurrent stable order of unfoldment in which a certain form undergoing regular changes manifests again and again, but so rapidly, that it appears to be in continuous existence (Bohm 1980: 194).

Finally, in their 1993 book *Undivided Universe* Bohm and Hiley, when discussing quantum field theory and emphasising the ontological primacy of movement required by relativity, summarize this non-individualistic line of thought as follows:

...the essential qualities of fields exist only in their movement [...] The notion of a permanently extant entity with a given identity, whether this be a particle or anything else, is ... at best an approximation holding only in suitable limiting cases. (1993: 357).

Thus much of Bohm's work supports the idea that individuals are not metaphysically fundamental in the light of contemporary physics. Bohm's emphasis on notions such as "structure process" (1965b), "order" and "movement" (1980) as fundamental in physics, suggests that the philosophical home of Bohm (and Hiley's) general approach to physics might well be found in some form of scientific structuralism which takes movement as fundamental, rather than in a metaphysics which takes individuals as basic (cf. Ladyman and Ross 2007).

However, as is well known, just after completing his 1951 text-book, Bohm discovered an interpretation of quantum theory which seems to give individuals a much stronger status than the usual interpretation of quantum theory. While acknowledging the wholeness of quantum phenomena, Bohm had grown dissatisfied with the usual "Copenhagen" interpretation. In particular, he felt this interpretation failed to address what is taking place between measurements, and after discussions with Einstein, began to look for an alternative (for his own account of this see Bohm 1987).

In 1952 he published two papers that proposed an interpretation of quantum theory in terms of "hidden variables". This interpretation provides a hypothetical description of individual quantum systems. It may help to begin to see the relevance of Bohm's 1952 interpretation to the question of individuality if we consider how he contrasts his approach with that of Bohr. Bohm writes:

...Bohr suggests that at the atomic level we must renounce our hitherto successful practice of conceiving of an individual system as a unified and precisely definable whole, all of whose aspects are, in a manner of speaking, simultaneously and unambiguously accessible to our conceptual gaze. ... in Bohr's point of view, the wave function is in no sense a conceptual model of an individual system, since it is not in a precise (one-to-one) correspondence with the behavior of this system, but only in a statistical correspondence (1952a: 167-8).

In contrast to this, Bohm's alternative interpretation regards

...the wave function of an individual electron as a mathematical representation of an objectively real field (1952a: 170).

Thus for Bohm, an individual quantum-mechanical system has two aspects:

it is a synthesis of a precisely definable particle and a precisely definable ψ -field which exerts a force on this particle (1952b: 188).

In this particular early approach, it is assumed that the particle has a well-defined position and momentum at all times and it thus moves along a trajectory. Actually, Bohm had independently rediscovered and made more coherent Louis de Broglie's 1927 pilot-wave model; this interpretation has thus also been called the de Broglie-Bohm theory. Other names which denote different versions of the approach, include "the causal interpretation", "Bohmian mechanics" (see Goldstein 2013) and "the ontological interpretation" (Bohm and Hiley 1993). In this paper we refer to this approach as "the Bohm theory" and follow Bohm's own preferred way of thinking about the approach which gives an important role to the so called quantum potential energy.

Now, if the Bohm theory is a coherent option, it undermines the arguments of those who claim that non-relativistic quantum theory somehow forces us to give up the notion that quantum objects are individuals with well-defined identity conditions. Ironically, as the above quotes show, there is clearly also a tension between the Bohm theory and much of Bohm's own other more structuralist and process-oriented work - both before and after 1952.

The question of whether quantum particles are individuals is also raised by the philosophers James Ladyman and Don Ross in their thought-provoking and important book *Every Thing Must Go* (2007, hereafter ETMG). They advocate the view that quantum particles are not individuals, (or, at most, are weakly discernible individuals). They acknowledge that there seem to be individuals in the Bohm theory, but, on the basis of Brown *et al.*'s (1996) discussion, judge these to involve *haecceities* and imply that such Bohmian individuals can be dismissed as idle metaphysics.

In this paper we shall be focusing on the 1952 approach and its later developments, as well as with Ladyman and Ross's evaluation of its relevance to the question of individuality of quantum objects. We suggest that the Bohm theory shows that there is room for individuals in at least non-relativistic quantum theory, albeit these "Bohmian individuals" are very different from simple classical particles. However, it is important to keep in mind that Bohm and Hiley's more general implicate order approach (which seeks to unite quantum theory and relativity) does not give individuals such a strong status and is in some ways similar to Ladyman and Ross's structuralist approach. Thus, although we will below criticise Ladyman and Ross's reading

of the non-relativistic Bohm theory, we think that their overall structuralist approach seems fruitful when seeking a more general theory (such as the implicate order) which can reconcile quantum theory and relativity.

In what follows we will first give a brief account of Ladyman and Ross's views on individuality in quantum mechanics (section 2). We then introduce the Bohm theory, focusing on the way it seems to have room for quantum individuals (section 3). In section 4 we first present Ladyman and Ross's criticism of Bohmian individuality and then go on to challenge this criticism by drawing attention to the fact that Brown *et al* (1996, 1999) suggest that in the Bohm theory, it is not unreasonable to assume that properties such as mass and charge also reside in the particles (cf. French 2011: 14; French and Krause 2006: 174). We then illustrate the nature of Bohmian individuals by considering the transition from quantum to classical behavior and the explanation of the AB-effect in terms of the quantum potential (section 5). In section 6 we try to obtain a deeper mathematical insight into the issue by considering the Bohm theory in the light of new mathematical developments in symplectic geometry due to de Gosson and Hiley. This discussion examines more closely the relation between the classical Hamilton-Jacobi equation and its quantum counterpart, equation (3) below. We find a much closer relation between these two mathematical structures than is generally recognised and argue that this has a direct consequence for the question of individuality. In conclusion, we suggest that in non-relativistic quantum theory, whether or not particles are individuals is a genuinely open question in a stronger sense than Ladyman and Ross imply. However, we note also that Bohmian individuals can best be understood in terms of notions such as "structure process", which suggests an affinity to Ladyman and Ross's structuralist approach.

2 Ladyman and Ross on individuality in quantum mechanics.

In the third chapter of ETMG, Ladyman and Ross discuss identity and individuality in quantum mechanics. Following French and Redhead (1988), they first establish that indistinguishable elementary particles, that is particles that have the same mass, charge etc., behave differently in quantum mechanics than they do in classical statistical mechanics. For quantum particles an 'indistinguishability postulate' states that a permutation of indistinguishable particles is not observable and thus those states which differ only by a permutation of such particles are treated as the same state with

a different labelling. This might point to the view that quantum particles are not individuals.

Individuality is however an ontological property whereas (in)distinguishability is an epistemic one. So how are these two related? Ladyman and Ross identify three candidates in the philosophical tradition for individuality:

1. transcendent individuality: the individuality of something is a feature of it over and above all its qualitative properties;
2. spatio-temporal location or trajectory;
3. all or some restricted set of their properties (the bundle theory) (ETMG, p. 134).

#1 above is ruled out because it involves *haecceities*, and thus involves what Ladyman and Ross would consider idle metaphysical speculation. Granting this restriction for the sake of the argument, the interesting candidates are #2 and #3

A connection between individuality and distinguishability is given by the Principle of the Identity of Indiscernibles (PII), which can be taken roughly to state that no two objects have exactly the same properties. It is easy to see that everyday objects satisfy both #2 and #3, while the point particles of classical mechanics satisfy #2. Then for both everyday objects and particles of classical mechanics PII is true and individuality and distinguishability can be taken to be the same thing. However, for certain quantum systems neither #2 nor #3 seems to hold. Ladyman and Ross take as an example of such a state the singlet state of two electrons orbiting a helium atom:

$$\psi = 1/\sqrt{2}[|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2] \quad (1)$$

Here any property that can be ascribed to particle 1 can also be ascribed to particle 2. So in this state the two electrons share all their extrinsic and intrinsic properties, thus falling foul of both #2 and #3. So it would seem that quantum particles are not individuals.

However, this conclusion might follow from a too strict a notion of discernibility. Following Saunders (2003a, 2003b, 2006), Ladyman and Ross give three notions of discernibility:

- (i) absolute discernibility
- (ii) relative discernibility and

(iii) weak discernibility.

These can be defined as follows (ETMG, p 137):

- (i) “Two objects are absolutely discernible if there exists a formula in one variable which is true of one object and not the other”. This holds for ordinary everyday objects.
- (ii) “Two objects are relatively discernible just in case there is a formula in two free variables which applies to them in one order only. ...[T]he points of a one-dimensional space with an ordering relation, since, for any such pair of points x and y , if they are not the same point then either $x > y$ or $x < y$ but not both”.
- (iii) “Two objects are weakly discernible just in case there is two-place irreflexive relation that they satisfy.” The Fermions in a singlet state are discernible in this sense, as they satisfy the relation ‘is of opposite spin to’.

Now since electrons in the singlet state are discernible they can be viewed as individuals. But they are weakly discernible. This is a thoroughly structuralist view “...as individuals are nothing over and above the nexus of relations in which they stand.” (ETMG, p. 138.)

3 The Bohm theory

Now let us turn to consider how the Bohm theory deals with these situations. It is assumed that every particle has a well-defined position and momentum and is accompanied by a field ψ which satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r})\psi \quad (2)$$

If we make a polar substitution

$$\psi(\mathbf{r}, t) = R(\mathbf{r}, t) \exp[iS(\mathbf{r}, t)/\hbar]$$

and then separate out the real and imaginary parts, we find two equations, firstly

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + Q + V = 0 \quad (3)$$

where

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \quad (4)$$

is known as the quantum potential. The second equation

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m} \right) = 0 \quad (5)$$

is a probability conservation equation. We also identify

$$\mathbf{p} = \nabla S.$$

This is known as the guidance condition, from which the trajectory of the particle can be calculated. Figures 1 and 2 provide well-known visualizations.

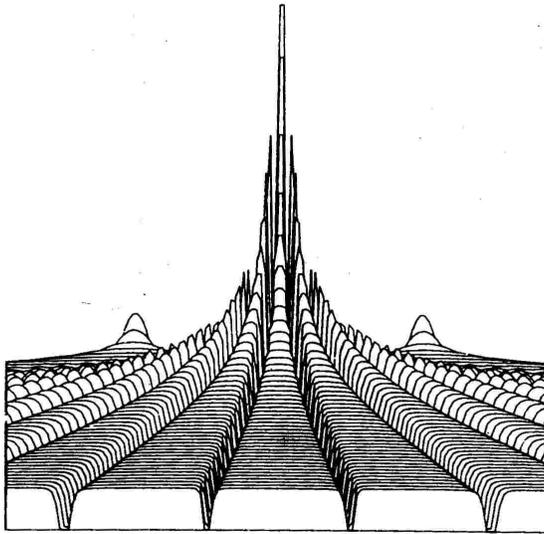


Figure 1: Quantum potential for two Gaussian slits

So there is a version of quantum theory (the Bohm theory) according to which each particle has a definite and distinct trajectory at all times. This suggests that quantum particles are individuals, with position being the property in virtue of which particles are always different from one another.

The biggest problem for retaining the notion of individuality is particles in entangled states described by equation (1). This is an entangled spin

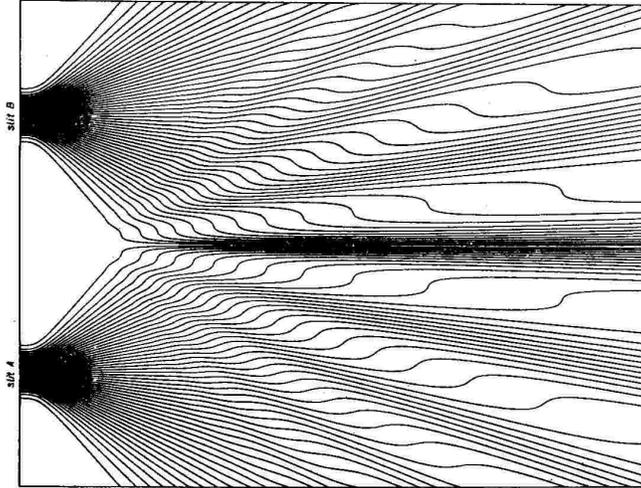


Figure 2: Trajectories for two Gaussian slits

state which has been discussed in detail by Dewdney, Holland, Kyprianidis and Vigier (1988), but for our purposes here it is sufficient to consider a general two-body wave function, $\psi(\mathbf{r}_1, \mathbf{r}_2, t)$ and use the two-body Schrödinger equation to find

$$\frac{\partial S}{\partial t} + \frac{(\nabla_1 S)^2}{2m_1} + \frac{(\nabla_2 S)^2}{2m_2} + Q(\mathbf{r}_1, \mathbf{r}_2, t) + V(\mathbf{r}_1, \mathbf{r}_2) = 0 \quad (6)$$

The second and third terms in this equation correspond to the kinetic energies of each particle so once again the problem of individuality does not seem to arise in the Bohm theory. The information about the entanglement is encoded in the non-local quantum potential energy term $Q(\mathbf{r}_1, \mathbf{r}_2, t)$. Furthermore since the trajectories do not cross, we follow Brown, Sjöqvist and Bacciagaluppi (1999: 233) and conclude that indistinguishable fermions will always have distinct trajectories (for further discussion see French 2011: 14-15; French and Krause 2006: 178). Thus individuality is preserved.

4 The Bohm theory and *haecceities*

So the Bohm theory seems to suggest, *contra* Ladyman and Ross, that quantum particles can be individuals in a stronger sense than they claim. They do acknowledge the existence of the Bohm theory in a footnote, but do not see it as a problem for their non-individualistic view. They write:

Of course, there is a version of quantum theory, namely Bohm theory, according to which QM is not complete and particles do have definite trajectories at all times. However, Harvey Brown *et al.* (1996) argue that the ‘particles’ of Bohm theory are not those of classical mechanics. The dynamics of the theory are such that the properties, like mass, charge, and so on, normally associated with particles are in fact inherent in the quantum field and not in the particles. It seems that the particles only have position. We may be happy that trajectories are enough to individuate particles in Bohm theory, but what will distinguish an ‘empty’ trajectory from an ‘occupied’ one? Since none of the physical properties ascribed to the particle will actually inhere in points of the trajectory, giving content to the claim that there is actually a ‘particle’ there would seem to require some notion of the raw stuff of the particle; in other words *haecceities* seem to be needed for the individuality of particles of Bohm theory too. (ETMG, p. 136 fn.)

Note first that as we have already indicated, Ladyman and Ross are somewhat one-sided in reporting the views of Brown, Elby and Weingard (1996). For in their paper Brown *et al.* are *not* arguing for the view that in the Bohm theory, properties like mass, charge, and so on, normally associated with particles are *only* inherent in the ψ -field and not in the particles. What they do argue for is that certain experiments (for example, certain types of interferometry experiments) rule out the possibility that these properties are associated with the Bohm particle *alone*. They point out that there are two principles we can adopt here. Firstly, there is the *principle of generosity*, according to which the properties can be attributed to both the ψ -field and the particle.

Secondly, there is the *principle of parsimony* according to which properties such as mass are attributes not of the particle but of the ψ -field alone. They do not take a definite stand on which principle we should adopt. However, they draw attention to reasons to adopt the principle of generosity, while at the same time indicating difficulties inherent in the principle of parsimony. It thus seems clear that, *contra* what Ladyman and Ross suggest, they are more in favor of the principle of generosity. To be fair, however, we should acknowledge that the issue is subtle and people’s views on this vary— it seems that Harvey Brown himself was in favor of parsimony before opting for generosity. For Brown *et al.* (1996) acknowledge that the principle of parsimony is implicit in Brown’s earlier work; they also note that it

is implied by some of Bell's suggestions (Bell 1990, 30).

In this paper we reject the principle of parsimony and argue for a particular version of the principle of generosity. Note especially that we are not arguing that Bohmian individuals are mere particles. Instead, we propose that the particle and the ψ -field should not be considered as separate entities, and thus a Bohmian individual is something that has both of these as two different aspects – but it is still an individual. This sort of idea goes back to at least Bohm's 1957 book *Causality and Chance in Modern Physics*, where he wrote:

...our model in which wave and particle are regarded as basically different entities, which interact in a way that is not essential to their modes of being, does not seem very plausible. The fact that wave and particle are never found separately suggests instead that they are both different aspects of some fundamentally new kind of entity which is likely to be quite different from a simple wave or a simple particle, but which leads to these two limiting manifestations as approximations that are valid under appropriate conditions. (1957/1984: 80)

Note especially that, according to the 1957 Bohm, the ontological message of quantum theory is not that quantum objects are not individuals; it is rather that they are fundamentally new kind of individuals.

So, are Bohmian particles bare? Let us go into this question in greater detail. Brown, Dewdney and Horton (1995) introduce and define the localized particle properties thesis (LPP): particle properties (such as mass, charge etc.) are attributes of the particle rather than the ψ -field. That is the mass, say, of the particle is localized at the position of the particle at all times. They go on to point out that several experiments seem to violate the LPP.

In neutron interferometry experiments of Colella, Overhauser, and Werner, (1975), the 'shifted' interference pattern created by a neutron stream traveling through a beam splitter along two routes, one with at higher gravitational potential than the other. According to the Bohm theory the particle travels one of the paths while the ψ -field travels both paths. Brown *et al* note that if we assume that all of the electron's gravitational mass is concentrated in the path where the particle is, it becomes difficult to understand intuitively why the interference pattern is shifted. For if the empty path ψ -field carries no gravitational mass, how could the difference in the gravitational potential integrated over the two paths be felt by the particle? So they argue that for gravitational mass, the LPP seems to be violated.

In the Aharonov-Bohm effect a similar thing happens for charge, as noted by Brown, Sjöqvist and Bacciagaluppi (1999, p. 234 fn):

The expression for the phase shift due to the flux in the shielded solenoid depends on the electrons charge being present on spatial loops within the support of the wave function and enclosing the solenoid.

But again the trajectory of the Bohmian particle associated with the charged particle does not encircle the solenoid. So the LPP seems to be violated for charge.

It is clear that LPP was motivated by the classical concept of a particle. But why should we expect a ‘quantum particle’ to be a simple ‘classical particle’? Already Weyl (1924) has warned us:

Hence a particle itself is not even a point in field space, it is nothing spatial (extended) at all. However, it is confined to a spatial neighbourhood, from which its field effects originate.

He even goes as far to call the concentrated region of field, an “energy-knot” which propagates in empty space just as a water wave advances over the surface of the sea.

We noted above how Bohm himself wrote in 1957 that the notion of a simple particle in his model is an approximation. But this leaves us with the question of what is the nature of the “fundamentally new kind of entity” that is the true Bohmian individual.

5 Bohmian individuals

To obtain a better understanding of the nature of Bohmian individuals, let us return to examine the context in which quantum theory arose in the first place in more detail. In order to do this we must first re-examine classical dynamics in the context of the Hamilton-Jacobi (HJ) theory. It should be remembered that this theory was developed at the beginning of the 19th century for utilitarian reasons, namely, to make it easier to solve Hamilton’s equation of motion for planetary systems. However it was soon realised that it provided an overall unifying description of classical mechanics emphasising that symplectic symmetry that lies at the heart of classical kinematics.

The solution of the classical Hamilton-Jacobi equation gives an ensemble of possible trajectories, but of course, the particle takes only one of these trajectories. Which trajectory it takes depends on the initial position of the

particle i.e., where it actually was at $t = 0$. Thus it is the presence of the particle that accounts for which trajectory is actually taken. If you have this initial information, you will be able to predict at which point a particle will hit, say, a screen placed perpendicular to the expected trajectories.

In the Bohm approach (Bohm and Hiley 1993), attention is focussed on the quantum version of the Hamilton-Jacobi equation (3). This equation will also give an ensemble of trajectories, but these trajectories will, of course, differ from those derived from a classical HJ equation simply because of the presence of the additional term which has been called the quantum potential energy (QPE).

For the time being, let us retain the notion that the particle has simultaneously a precise position and momentum at all times, then its position at some initial time t_i will determine which trajectory it follows. But in the quantum case there is, of course, a difference from the classical case. In the latter we can, in principle, ‘place’ a particle at any initial point with any initial momentum we care to choose, so that we, ourselves, can choose which trajectory the particle will take.

This option is not open to us for the quantum particle. We cannot control the initial position and momentum simultaneously, so that we cannot ‘place’ the particle at a given position of our choice, together with a pre-determined momentum. This is the message of the Heisenberg uncertainty principle. Nevertheless it is the key assumption in the Bohm model that the particle always has a simultaneously well-defined position and momentum, even though we do not know nor can we control what these values actually are. To produce our particle experimentally, we have to rely on sources such as a furnace, or an electron gun or even a quantum dot, sources that can only produce a distribution of initial conditions. Each particle in that ensemble will have a definite initial position and momentum as it enters the experiment and it is those values that determine the particular trajectory it will follow. The fact that we cannot control this initial position is irrelevant to the hypothesis that it actually has a definite position and momentum.

It is clear that different experimental arrangements will produce different ensembles of trajectories and this difference is encoded in the QPE. The solution of Schrödinger’s equation will determine the form of the QPE. Suppose we have a situation in which the quantum potential is time dependent and actually becomes smaller as time progresses, then we can show that the ensemble of trajectories merge smoothly into the classical ensemble of trajectories. Thus a particle following a trajectory in the quantum domain will become a particle obeying the rules expected of a classical particle.

A simple model illustrating this merger was presented by Hiley and Mufti

(1995). In our view these results present strong evidence that it is a coherent possibility that a particle keeps its identity and individuality in a quantum context, even though some of its energy is involved exploring its environmental neighbourhood. According to this model the whole process is an individual coherent process, but notice it need no longer be a point-like process. This means that the notion of individuality in the quantum domain is very different from the way it is understood classically (as indeed anticipated by Bohm in his remarks quoted above (1957:88)). Moreover, this is something that Niels Bohr (1958), too, noticed, a point that we will discuss in more detail later.

With this in mind, let us now turn to examine the properties of the quantum potential in the experimental situation that involves the AB effect. Because of the intriguing questions raised by the AB effect, Philippidis, Bohm and Kaye (1982) decided to calculate the quantum potential and the trajectories in the AB experiment. The result for the quantum potential is shown in figure 3.

If we compare this QPE with that shown in figure 1, where no flux is present, we see that the pattern of the quantum potential has been shifted off the axis of symmetry due to the presence of the enclosed flux. This, in turn, produces a shift in the ensemble of trajectories as was shown in Philippidis, Bohm and Kaye (1982). This resulted in an overall pattern shift which has actually been observed in experiment (see Bayh (1962)). Note that a similar result will apply to the gravitational case. From these two examples we see that the form of the quantum potential energy depends on the whole experimental arrangement and not merely on the local interactions with the classical fields. Bohr (1958) famously emphasized the importance of the experimental arrangement. By using the real part of the Schrödinger equation which includes the quantum potential, we see clearly why the experimental conditions are a vital feature of quantum processes, thus supporting Bohr's important point.

6 The Role of the Hamilton-Jacobi Theory

The way we have derived the quantum Hamilton-Jacobi equation (3) hides a much deeper relation between classical and quantum dynamics. Mathematically, as we have already remarked, they have a common kinematic symmetry, namely, the symplectic group of transformations. As Melvin Brown (2006: v) has succinctly put it,

[t]his very general group of transformations maintains the fun-

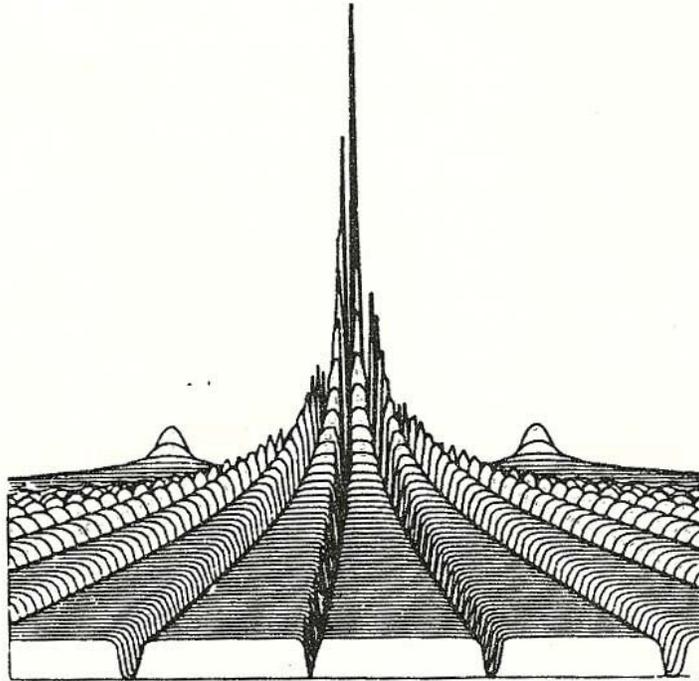


Figure 3: The quantum potential for the Aharanov-Bohm effect. Notice the asymmetrical shift

damental relationship between position and momentum in mechanics, and its covering group (the metaplectic group) correspondingly transforms the wave function in quantum mechanics.

Assuming that the reader may not be familiar with the mathematics, we will here approach the subject in a non-formal way (for a more technical presentation, see e.g. de Gosson 2001; Brown 2006). Notice that unfortunately we do not have a method of calculating the QPE or the trajectories independently of the Schrödinger equation. However given the QPE, the steps in solving the QHJ are identical to the steps in solving the classical HJ equation.

What then is the significance of this relationship? To answer this question let us consider in more detail the classical Hamilton-Jacobi theory. As we have already remarked initially this theory was thought of as a useful tool for solving Hamilton's equations of motion because, by a judicious choice of

canonical (symplectic) transformations, these equations can be reduced to a form that can be more easily solved.

However Hamilton himself saw the theory in a different way. He noticed that rays in optics could be described by Hamilton's equation of motion by identifying p with the angle of the ray to the optic axis. (See Guillemin and Sternberg 1984). What about the wave properties? Hamilton suggested that the surfaces of constant action were some form of wave fronts with the rays (particle trajectories) perpendicular to these surfaces so that the electromagnetic field energy could be regarded as flowing along the rays. Notice in this way the energy is merely a concentration of field energy which was eventually identified with the photon. Suggestive as this was, Hamilton's ideas were never compelling.

With the discovery of the wave properties of electrons and atoms, Schrödinger again took up Hamilton's ideas. In his second paper, which immediately followed the one containing the announcement of what we now call the Schrödinger equation, he notes that "classical mechanics is a complete analogue of geometric optics", and then adds, "Then it is the case of seeking an "undulatory mechanics"—and the most obvious route to this is in fact the form of the Hamiltonian picture based on the wave theory" (Schrödinger 1926). Schrödinger's arguments to derive his equation can at best be heuristic. Even Schrödinger himself, in the course of his derivation writes "I realise that this formulation is not quite unambiguous". But this should not be surprising as the necessary mathematics of symplectic geometry did not exist at the time this work was being done.

However the equation quickly gave results that agreed with experiment, so that the equation was taken as an *a posteriori* given, independent of its origins. It became *the* defining equation of quantum phenomena and as a consequence of trying to understand this equation, the 'wave function' became the centre of attention. With this position came the paradoxes that remain unresolved. Relatively few physicists or philosophers have attempted a sustained exploration of the deeper mathematical background from which the equation appears. Indeed the Schrödinger equation is taken as a given, arising as if by magic. Even Feynman *et al* (1963) acknowledges that the equation was not derived from anything known in physics or mathematics. As he remarked: "It came out of the mind of Schrödinger".

Since the first appearance of the Feynman lectures, big advances have been made in the mathematical analysis of Hamilton dynamics, the Hamilton-Jacobi theory and the 'undulatory mechanics' Schrödinger was looking for (see de Gosson 2001 and 2010 for a readable comprehensive treatment of the subject). This new work shows that there is a deep mathematical connection

between Hamiltonian dynamics and Schrödinger dynamics.

The connection begins to emerge as we examine the common symmetry, the symplectic symmetry, underlying both Hamiltonian dynamics and the Schrödinger dynamics. These deeper connections have emerged relatively recently in the mathematics literature and are just beginning to become appreciated by the physics community. What one learns is that classical mechanics and ray optics emerge at the level of the group itself, but wave theory and quantum mechanics begins to emerge at the level of the double cover of the group, the metaplectic group and its generalisation.

The importance of the double cover is a vital part of quantum mechanics. We are already very familiar with this idea in the case of rotational symmetry. Here the double cover of the rotation group is the spin group. This gives us the spinor with which we describe fermions and these spinors form the mathematical basis of the entangled singlet state given equation (1) above. Furthermore it is the relation between the group and its cover that gives us the explanation of the experimentally confirmed difference between a 2π and a 4π rotation that shows up with fermions (see Werner et al. 1975).

The symplectic group then is the key to the dynamics. Hamiltons equations of motion are invariant under a symplectic transformation (this transformation is traditionally known as a canonical transformation). What is not so well known is how the double cover provides the link between classical and quantum dynamics. To bring this out one needs to study symplectic geometry and it is this geometry that provides new insights into the relation between classical and quantum dynamics (see de Gosson 2001).

This is not the place to go into mathematical details and we will simply point out some interesting results to bring out the unexpected connection between classical and quantum mechanics. Firstly de Gosson and Hiley (2011) have shown that if we formally “lift” a Hamiltonian flow onto the double cover space, we find a unique flow in this covering space and this flow satisfies a Schrödinger-like equation. It is ‘Schrödinger-like’ because at this stage it contains an arbitrary parameter with dimensions of action to enable us to put position and momentum on the same footing. The mathematics alone does not enable us to identify this parameter with Planck’s constant. Its value is determined by experiment as it is in the standard theory.

Secondly de Gosson (2010) draws attention to a deep topological theorem in symplectic geometry known as the ‘Gromov no-squeezing’ theorem. This states that even in classical mechanics, it is not possible to reduce a canonical volume such as $\Delta x \Delta p$ by means of a Hamiltonian flow alone. When this region is lifted into the covering space it provides the source of

the uncertainty principle. In this way one can say that the symplectic features of the classical world contains the footprint of the uncertainty principle even at the classical level. This structure provides a rigorous mathematical background for Schrödinger's 'undulatory mechanics'.

Let us follow up this relation between the Hamiltonian and Schrödinger flows, as the time evolutions are called. The use of the term 'flow' is very appropriate here because we are talking about the energy carried by their respective fields; classically we have the motion generated by the action field $S(x, t)$ while in the quantum process we have two real fields, R and S where S can now be interpreted as the phase in the wave picture. In the classical limit R = a constant, while the phase S becomes the action, linking with Hamilton's original ideas.

Now we must consider how the individual, our central concern in this paper, enters the picture. We have already seen how Hamilton considered the optic ray as an element of the excitation energy of the field itself, a forerunner of what we now call the photon. Thus when we generalise to the Schrödinger case, we can regard the 'trajectory' as the locus of mean of some invariant feature of the energy carried by the two real fields. In this view the generating fields allows one to calculate the behaviour of a particular 'particle' that happens to be at a given point in space and time. There is no separate 'rock-like particle' that needs to explore the whole of space. It is the dynamics and its effective neighbourhood that determine the behaviour of the individual.

Why do we say 'the individual'? For this we need to consider an actual experiment. It is now possible to find sources that produce a single particle at a time, its dynamics is determined either by the Hamiltonian flow or by the Schrödinger flow depending on the experimental set up. The individual, its centre of energy then follows a particular trajectory defined by its initial condition. In both the classical and quantum cases there is only one individual. Indeed one can show that as the quantum potential energy becomes negligible the quantum 'particle' becomes the classical particle. In other words any interior structure of the particle can be neglected.

So far we have emphasised the similarity between the two flows. However it is obvious that the flows are very different but in precisely what way are they different? Let us start with a simpler question for which there is a well-defined answer. Let us ask "What is the difference between the group and its two-fold cover?" Locally they have the same Lie algebra, this algebra defines the group in the neighbourhood of the identity. Thus in a very small region about a given point, both flows are identical (see de Gosson and Hiley 2013 for a rigorous proof.)

The difference arises from their global properties, namely, in the global topology of the group manifolds. It is these global properties that distinguish between the symplectic and metaplectic groups. Thus lifting the Hamiltonian flow onto the covering group means modifying the flow so that it responds to the global properties of the metaplectic group. In other words the Schrödinger flow must take account of this global topology and must be able to distinguish sets of paths that belong to different elements of the holonomy group.

In the case of the AB effect, the flow must reflect the difference between those closed paths that encircle the enclosed flux and those that do not. What this means is that the Schrödinger flow itself is a global flow, not a local flow. When we analyse this flow in terms of the Hamilton-Jacobi theory we find these global features are encoded in the quantum potential. In this sense the QPE is not the source of a force acting on the particle. Rather it is a potentiality for the behaviour of the particle-like process that finds itself at a particular region in space. Then its kinetic energy, and therefore momentum, is such as to satisfy the quantum HJ equation. In this way the particle appears to be responding to a ‘force’ whereas in fact the particle is following a trajectory defined by a global flow. Thus there is just one individual particle whose evolution is determined by the Schrödinger flow.

In the AB situation, an ensemble of incident particles would then give rise to a shifted interference pattern, the shift being determined by the enclosed magnetic flux. Nowhere do we lose the identity of the individual particle. This notion of individuality is strengthened when it is realised that if the particle were not charged, then its Schrödinger evolution would be very different and the fringe pattern would be unaffected by the presence of any enclosed flux. In this sense the quantum potential energy is a ‘private’ energy; it is ‘individual’ in the sense of belonging to the individual particle. Note that this is an example where quantum theory seems to involve stronger and more peculiar individuality than classical physics! This point often left unnoticed in discussions of individuality in the quantum theory where one typically emphasizes the non-individualistic aspects of the quantum domain (see however Brown *et al.*1996: 313-4). If two particles are conventionally described by a product of two wave functions and the particles do not interact through a classical potential, they do not experience each others quantum potential even though they may both be in a region of space where their factor wave functions have significant spatial overlap.

The radically different nature of individuality in the quantum domain becomes more apparent when two or more particles become ‘entangled’ as illustrated in equation (1). In the Bohm approach these particles are coupled

by a common quantum potential energy. This potential is non-local in the sense that the behaviour of one particle is ‘locked’, as it were, into the behaviour of the other. In this case the Schrödinger flow involves both particles, giving a time evolution that involves the two spatially separated particles behaving as a single entity if uninterrupted. It is tempting to see such an entity as new type of individual, where we find a “twoness” in an underlying individual whole. This is in contrast to two particles described by a product of two wave functions, which can be seen as separate individuals, as noted above.

If we return to examine the details of a pair of particles described by equation (1), we can solve the two-body QHJ equation and find an ensemble of correlated trajectories as has been shown by Dewdney *et al.* (1986, 1987, 1988). If one of the particles enters the field of a Stern-Gerlach magnet, it is then deflected either ‘up’ or ‘down’ depending on the positions of each particle at the time just before the particle enters the magnetic field. The particle in the field has its trajectories changed while the other particle continues in a straight line. This is a dramatic illustration of the non-local effect of the quantum potential energy. But in spite of this non-local effect the ensemble of trajectories is still determined by the QHJ.

The spin properties of the individual particles undergo some interesting changes in this situation. In the initial singlet state, the individual particles, although possessing spin-half, do not possess any component of spin. This is consistent with the fact that the state of the pair has net spin zero, i.e. $S = 0, S_z = 0$. This is a surprising result, but shows quite clearly that the individual parts cannot be thought of as ‘little spinning spheres’, a point that was emphasised by Weyl (1931).

However as one of the pair passes through the Stern-Gerlach magnetic field, the quantum potential energy changes and this change manifests itself in the following way. The particle in the inhomogenous magnetic field develops a z -component of the spin, while its partner develops the opposite z -component in such a way that the total angular momentum is always conserved.

Please note that this remarkable behaviour follows from the mathematics, from the QHJ which itself is a direct consequence of the Schrödinger flow. To reach this stage we do not have to introduce any physical speculation as to why the Schrödinger flow takes the form it does. Rather the Schrödinger flow is a direct consequence of lifting the classical Hamiltonian flow on the double cover space to exploit the global topological properties of the covering group.

One still might not be satisfied by this explanation. But then we are faced with the question “How do particles with spin one-half exhibit their $2\pi - 4\pi$ behaviour?” Again the only answer we have is that quantum processes exploit the properties of the double cover of the rotation group, the spin group or more formally, the Clifford group. If we took the words of Bohr seriously, we would expect the flow to be determined by the overall experimental conditions. Bohr writes that it is impossible to make a

... sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear (Bohr 1958: 39-40).

This, according to Bohr, means that we must give up any way of trying to ‘picture’ what is going on but what he has not supplied is an adequate reason as to why it is necessary to adopt such an extreme position and to further claim there exists no other possibility. Indeed because of this, commentators of the Bohr view have called it a ‘non-interpretation’ (Leggett 1982) or anti-realist (Faye 1991).

Our proposal is that it is the structural properties of the key symmetries of motion and of space that necessitate such a position. This supplies a structural *raison d’etre* for adopting this position. Of course one might not be happy with leaving it as a mathematical theorem and would like some underlying physical processes that necessitate the use of the covering group structure. That is what Bohm (1980), Bohm and Hiley (1993, ch 15) and Hiley (2011) have been trying to do by taking process as fundamental, but this is another story that we will not go into here.

7 Concluding remarks.

We have above sketched a possible way one could understand the status of individuals in the Bohm theory, in the light of recent explorations of the underlying mathematical structures. It seems that the prospects of individuality in the Bohm theory are stronger than Ladyman and Ross imply. This suggests that there is an underdetermination of metaphysics by physics in non-relativistic quantum theory when it comes to the question of individuality (as indeed has been emphasized by French and Krause 2006: 189-197). However, it is important to realize that the notion of an individual in the Bohm theory is very different from what we would expect from the classical perspective. For although the Bohmian quantum individual has a well defined energy, that energy is not a local energy. This is consistent with Niels

Bohr's views in two ways. Firstly, as remarked above and re-emphasised in Bohm and Hiley (1993), the particle is never separated from the quantum field. It is an invariant feature of the total underlying process. This is consistent with Bohr's notion of the "impossibility of subdividing quantum phenomena" in the sense that the whole experimental arrangement must be taken into account (Bohr 1958:50-51).

Secondly, the individual is not a localised point-like object. As Bohr remarks (1958:73) the quantum process is a "closed indivisible phenomenon". The energy is not localised at a point. In fact complementarity can be taken to imply that energy transcends space-time. Nevertheless there is a centre of energy, a generalisation of the centre of mass which can be given a position in space-time. It is this centre that moves with the Bohm momentum.

These ideas are not consistent with a classical notion of a particle and we feel can only be given a proper meaning in terms of something like Bohm's (1965b) notion of "structural process". Thus the overall Bohmian approach to physics does not, from the metaphysical point of view, mean a return to the individuals of classical physics, but has strong structuralist features. In particular, and as was mentioned in the Introduction, Bohm and Hiley have since the 1960s been developing broader scheme they call "the implicate order", which goes beyond the 1952 Bohm theory (Bohm 1980; Bohm and Hiley 1993: ch15; Hiley 2011; Pylkkänen 2007; for Bohm's own attempt to reconcile "hidden variables" and the implicate order, see his 1987). We note that this scheme seems to have some relevant similarities to Ladyman and Ross's ontic structural realism, while there also may be some significant differences. The discussion of these similarities and differences will, however, be a subject of another study (some preliminary attempts have already been made by Pättiniemi 2011 and Pylkkänen 2012).

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